Stability of one-dimensional array solitons

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The array soliton stability in the discrete nonlinear Schrödinger equation with dispersion for periodic boundary conditions is studied. The linear growth rate dependence on the discrete wave number and soliton amplitude is calculated from the linearized eigenvalue problem using the variational method. In addition, the eigenvalue problem is solved numerically by shooting method and a good agreement with the analytical results is found. It is proved numerically that the results for the instability threshold for the circular array coincides with the quasicollapse threshold for the case of open arrays with initial pulses in a form of array solitons.

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I. INTRODUCTION

Wave instabilities are important physical phenomena typically arising in nonlinear physical systems. Mathematical modeling of these nonlinear systems with a different origin often leads to one of the universal nonlinear evolution equations, such as the nonlinear Schrödinger equation (NLS), Korteweg–de Vries equation, Sine Gordon equation, Kadomtsev-Petviashvili equation, etc. These nonlinear partial differential equations represent continuum models for different nonlinear systems exhibiting diverse and fascinating phenomena including solitons, pattern formation, collapse (blow-up) solutions, and spatiotemporal chaos, closely related with the wave instability phenomena. The problems of the soliton stability were extensively studied during the last forty years and still attract a large scientific interest (see review paper from Kivshar and Pelinovsky [1] and references therein).

On the other hand, the matter itself is discrete, i.e., it consists of many elementary entities, and in a situation when the spatial scale of the physical process approaches the size of the elementary entities, constituents of the physical system, a continuum approach fails and the discreteness of the system must be taken into account. In this case the mathematical modeling leads to one of the discrete versions of the nonlinear evolution equations. Discreteness introduces a number of features in the system dynamics concerning the solitons existence and their stability, suppression of the wave collapse phenomena, etc. One of the fundamental models describing dynamics of different nonlinear discrete systems is discrete nonlinear Schrödinger (DNLS) equation. For example, the energy transport in molecular chains of the α-helix structure of proteins [2], the propagation of nonlinear waves in discrete electrical lattices [3], DNA dynamics [4], and optical pulse propagation in nonlinear fiber arrays (NFA) [5] are all described with DNLS equation. Nonlinear fiber arrays attract a special attention due to their possible technological application in developing all-optical devices capable to compress, amplify, and switch optical pulses [6–8]. The central role in the theoretical description of the optical pulse propagation in NFA plays $1+2$ DNLS equation, with one discrete and two continuous variables, also known as continuum-discrete nonlinear Schrödinger (CDNLS) equation. Compared with the continuum two-dimensional (2D) NLS equation, the CDNLS equation exhibits features allowing an introduction of concepts in designing all-optical devices based on optical pulse propagation in NFA. One is the existence of multidimensional solitary wave solutions localized in both dimensions, discrete and continuous [9]. Another difference is the quasicollapse behavior of the CDNLS equation [7], closely related to the collapse phenomenon in 2D NLS equation. However, for CDNLS equation the collapse process, instead toward singularity, evolves to stable multidimensional solitary wave solution.

The problems of existence and stability of the solitary wave solutions in NFA localized in both continuous and discrete dimensions were considered in [7–12]. In particular, stability of continuous waves (CW) as well as 1D temporal soliton solutions under the restriction for the case where solutions have the same shape and phase in all waveguides was studied in [6,11,13]. The same stability problem but in a more general case for the moving CW and rotating solitons was studied in [14]. The authors in [6,11,14] have reported the conditions for the onset instability without details about the growth rate structure in the instability region. The aim of this paper is to give more detailed insight of the stability problem of 1D array soliton solutions with a complete growth rate dependence on the discrete wave number and soliton amplitude.

II. BASIC EQUATIONS

The CDNLS equation with one discrete and one continuous space variables reads

$$i\frac{\partial\psi_n}{\partial t}+\frac{\partial^2\psi_n}{\partial x^2}+2\psi_n|\psi_n|^2+(\psi_{n+1}+\psi_{n-1}-2\psi_n)=0,$$

$$n=2,3,\ldots,N-1. \tag{1}$$

The equations for the discrete elements 1 and N depend on the boundary conditions for the array. The closed (circularly arranged) array is described with the periodic boundary conditions $\psi_1=\psi_{N+1}$, while for the open array (linear array), when the elements ψ_1 and ψ_N are coupled only with one neighbor element, boundary conditions are given by $\psi_0=\psi_{N+1}=0$.

The CDNLS equation (1) is in a close connection with the two-dimensional continuum NLS equation

$$i\frac{\partial\psi}{\partial t}+\Delta\psi+2|\psi|^2\psi=0,\quad \Delta=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2},\tag{2}$$

and can be derived from it with the discretization of one of the variables in the 2D differential operator Δ, e.g., variable y. On the basis of this connection many analytical techniques originally developed for the continuous model (2) can be adopted for application on the discrete model (1). Moreover, comparison between discrete and corresponding continuous models allows extraction of useful information about the properties of the discrete model.

The variables in the CDNLS equation (1) have different meaning depending on the nature of the nonlinear continuous-discrete systems they describe. In particular, for the case of optical pulses propagation in NFA the CDNLS equation takes a form

$$i\frac{\partial\psi_n}{\partial t}+\beta\frac{\partial^2\psi_n}{\partial x^2}+2\gamma\psi_n|\psi_n|^2+\delta(\psi_{n+1}+\psi_{n-1}-2\psi_n)=0,\tag{3}$$

where ψ_n is the electric field envelope into the nth fiber, z is the distance along the fibers in the frame of reference moving with a group velocity v_g, $t=T-z/v_g$ is the retarded time, δ is the coupling coefficient between neighboring fibers, β is the group velocity dispersion parameter, and γ is the nonlinear coefficient. Taking into account that variables t and x in Eq. (1) have a reverse meaning in the fiber optics (t corresponds to z and x to t) and introducing dimensionless variables $\psi_n\to\psi_n\sqrt{\gamma/\delta}$, $t\to t\sqrt{\delta/\beta}$, and $z\to\delta z$, Eq. (3) becomes equivalent with the Eq. (1).

The CDNLS equation (1) is not integrable but possess conserved quantities such as the number of quanta (P) and Hamiltonian (H)

$$P=\sum_n\int_{-\infty}^{\infty}|\psi_n|^2dx,\tag{4}$$

$$H=\sum_n\int_{-\infty}^{\infty}(|\psi_n-\psi_{n-1}|^2+|\psi_{n,x}|^2-|\psi_n|^4)dx,\tag{5}$$

which plays an important role in studying dynamics of the CDNLS equation.

For the system with periodic boundary conditions the set of stationary one-dimensional (independent on index n) solutions has a form $\psi_{n0}=g(x)\exp i\lambda^2 t$. The first one is the uniform continuous wave solution with $g(x)=\lambda/\sqrt{2}$ and the second one is the soliton array solution

$$g(x)=\lambda/\cosh(\lambda x),\tag{6}$$

uniform along the discrete dimension and localized in the continuous dimension. The real parameter λ in both cases is related to the solution amplitude.

It was shown analytically and numerically that both solutions are unstable under the small perturbations along the discrete dimension (transverse perturbations) [11] and that the instability develops to its final state in a form of multidimensional solitons [9] localized in both dimensions, continuous and discrete. In the following section we will focus our study on the transversal instability of the soliton array solution (6).

III. STABILITY ANALYSIS

In order to study the stability problem of the soliton array solution (6) we introduce small perturbations in the system

$$\psi_n(x,t)=[g(x)+\delta\psi_n(x,t)]\exp i\lambda^2 t,\quad |\delta\psi_n(x,t)|\ll g(x),\tag{7}$$

in a form $\delta\psi_n(x,t)=\delta\psi(x,t)\cos(sn)$, where with $s=2\pi/N$ is introduced the discrete wave number.

After the substitution of Eq. (7) into Eq. (1), keeping only linear terms in $\delta\psi_n$ and splitting the perturbations into real and imaginary parts $\delta\psi_n=a+ib$, the following set of two ordinary differential equations is obtained

$$\frac{\partial b(x,t)}{\partial t}=-\hat{L}_+a(x,t),$$

$$\frac{\partial a(x,t)}{\partial t}=\hat{L}_-b(x,t),\tag{8}$$

where $\hat{L}_\pm$ are linear second order differential operators defined with

$$\hat{L}_+=-\frac{\partial^2}{\partial x^2}+\lambda^2-6g^2(x)+4\sin^2(s/2),\tag{9}$$

$$\hat{L}_-=-\frac{\partial^2}{\partial x^2}+\lambda^2-2g^2(x)+4\sin^2(s/2).\tag{10}$$

The stability analysis for the case of the uniform continuous wave solution is simple but for the case of the array soliton solution (6) is complicated due to the x dependence of the operators $\hat{L}_\pm$. For the further analysis it is convenient to express these operators in terms of the following Sturm-Liouville-type operators

$$\hat{S}_+=-\frac{\partial^2}{\partial y^2}+6\tanh^2(y),$$

$$\hat{S}_-=-\frac{\partial^2}{\partial y^2}+2\tanh^2(y).\tag{11}$$

Defining a new independent variable $y=x\lambda$, the operators $\hat{L}_\pm$ take a form

$$\hat{L}_+=\lambda^2(\hat{S}_++\mu-5),$$

$$\hat{L}_-=\lambda^2(\hat{S}_-+\mu-1),\tag{12}$$

where μ is defined with

$$\mu = \frac{4}{\lambda^2}\sin^2(s/2). \tag{13}$$

The Sturm-Liouville operators $\hat{S}_\pm$ are known as one-particle operators in quantum mechanics and their spectra are well known [15]. They are positive definite and possess only positive eigenvalues. The smallest eigenvalues $\sigma_\pm^{(0)}$ and corresponding eigenfunctions $\psi_\pm^{(0)}$ in the discrete spectra of the operators $\hat{S}_\pm$ are

$$\sigma_-^{(0)} = 1; \quad \psi_-^{(0)} = 1/\cosh(y),$$

$$\sigma_+^{(0)} = 2; \quad \psi_+^{(0)} = 1/\cosh^2(y). \tag{14}$$

It is important to point out that the discreteness of the system is incorporated into the structure of the operators (12) only through the defined parameter μ (13), which allows straightforward application of the stability analysis originally used for the corresponding continuous model.

A. Energy principle

As a first step in the stability analysis we will try to use an energy principle that was applied to the soliton stability problems in continuous models [16–18]. If we consider a square integrable function $\psi(y)$ (scalar product exists and is finite), then eigenfunctions $\psi^{(n)}(y)$ corresponding to discrete eigenvalues $\sigma^{(n)}$ of a self-adjoint operator $\hat{L}$ represent stationary values of the functional

$$F_{\hat{L}}(\psi) = \frac{\langle \psi|\hat{L}\psi\rangle}{\langle \psi|\psi\rangle}. \tag{15}$$

It means that the functional (15) has a vanishing first variation for the eigenfunctions $\psi^{(n)}(y)$ and if it has a lower bound than it corresponds to the smallest eigenvalue $\sigma^{(0)}$ of the operator $\hat{L}$

$$\inf\ F_{\hat{L}}(\psi) = \frac{\langle \psi^{(0)}|\hat{L}\psi^{(0)}\rangle}{\langle \psi^{(0)}|\psi^{(0)}\rangle} = \sigma^{(0)}. \tag{16}$$

For the operators $\hat{L}_\pm$ defined with Eq. (12) follows that the corresponding functionals in a form (15) are bounded from below,

$$\inf\ F_{\hat{L}_+} = \frac{\langle \psi^{(0)}|\hat{L}_+\psi^{(0)}\rangle}{\langle \psi^{(0)}|\psi^{(0)}\rangle} = \frac{\langle \psi^{(0)}|\hat{S}_+\psi^{(0)}\rangle}{\langle \psi^{(0)}|\psi^{(0)}\rangle} + \mu - 5$$
$$= \sigma_+^{(0)} + \mu - 5, \tag{17}$$

$$\inf\ F_{\hat{L}_-} = \frac{\langle \psi^{(0)}|\hat{L}_-\psi^{(0)}\rangle}{\langle \psi^{(0)}|\psi^{(0)}\rangle} = \frac{\langle \psi^{(0)}|\hat{S}_-\psi^{(0)}\rangle}{\langle \psi^{(0)}|\psi^{(0)}\rangle} + \mu - 1$$
$$= \sigma_-^{(0)} + \mu - 1, \tag{18}$$

where $\mu > 0$ implying $s \neq m\pi$, where $m = 0,1,\ldots$. Finally, replacing the smallest eigenvalues and corresponding eigenfunctions from Eq. (14) for the lower bounds of the functionals (17) and (18) we have

$$\inf\ F_{\hat{L}_+} = \mu - 3, \tag{19}$$

$$\inf\ F_{\hat{L}_-} = \mu. \tag{20}$$

According to the procedure for the corresponding continuous model described in detail in [18] the energy principle is applicable for $\mu > 0$ and the system is unstable for $0 < \mu < 3$ where $\hat{L}_+$ is indefinite, and stable for $\mu > 3$ where $\hat{L}_+$ is positive definite. For the soliton array solution these results lead to the following instability condition

$$\lambda^2 > \lambda_c^2 = \frac{4\sin^2(s/2)}{3}. \tag{21}$$

The result (21) coincides with the result obtained in [11] and for $N \to \infty$ leads to $\lambda_c \approx 7.255$. This value corresponds to the threshold power $\nu = \sqrt{P(\lambda_c)} \approx 2.69$ that lies between lower ($\nu_l \approx 2.34$) and upper ($\nu_u \approx 4.89$) bounds obtained with a rigorous mathematical treatment in [13].

B. Variational principle

In order to find out more details about the instability modes satisfying the instability condition (21) we can use the variational approach [18,19]. For the normal exponentially growing modes $a(t,y) = a(y)\exp\gamma t$; $b(t,y) = b(y)\exp\gamma t$ with the growth rate γ, the Eqs. (8) are transformed to the following eigenvalue equations:

$$\hat{L}_+ a(y) = -\Gamma b(y),$$

$$\hat{L}_- b(y) = \Gamma a(y), \tag{22}$$

where $\Gamma = \gamma/\lambda^2$ is the normalized growth rate. The Eq. (22) can be derived from the variation of the action

$$\delta S = \delta \int_{-\infty}^{\infty} \mathcal{L}(a, a_y, b, b_y, y)\,dy, \tag{23}$$

where the Lagrangian is given by

$$\mathcal{L} = \frac{1}{2}(a_y^2 + b_y^2) + \left[\frac{\mu+1}{2} - \frac{3}{\cosh^2(y)}\right]a^2$$
$$+ \left[-\frac{\mu+1}{2} + \frac{\Gamma}{\cosh^2(y)}\right]b^2 + \Gamma ab. \tag{24}$$

It is well known that the variational principle results critically depend on the choice of the test functions. A choice of good test functions is an essential step in the application of the variational principle and to undertake it successfully is necessary to obtain some qualitative information about the solution of the eigenvalue problem (22). Combining Eq. (22) we obtain the equation.

$$\hat{L}_+\hat{L}_- b(y) + \Gamma^2 b(y) = 0, \tag{25}$$

where a solution is an eigenfunction $b^{(0)}(x,t)$ corresponding to the least eigenvalue of the operator $\hat{L} = \hat{L}_+\hat{L}_-$ where the role of the eigenvalue plays the growth rate Γ^2. Operator $\hat{L}$ is a product of two second order differential operators that are invariant with respect to space reversal and consequently the eigenfunctions must have even or odd parity. Another useful information for finding a good test function is knowledge of the marginally stable solutions representing the intersection points of the dispersion curves $\Gamma^2(\mu)$ with μ axis. These solutions can be found from the Eq. (22) for $\Gamma = 0$

$$\hat{L}_+ a(y) = \hat{L}_- b(y) = 0. \tag{26}$$

For the local marginally states with the discrete eigenvalues two even solutions can be found for the cutoff values of the parameter μ

$$a(y) = 0, \quad b(y) = 1/\cosh(y) \quad \text{for } \mu = 0, \tag{27}$$

$$a(y) = 1/\cosh^2(y), \quad b(y) = 0 \quad \text{for } \mu = 3. \tag{28}$$

The simplest choice of the test functions $\tilde{a}(y)$ and $\tilde{b}(y)$ based on the previous consideration with two variational parameters α and β is

$$\tilde{a}(y) = \alpha/\cosh^2(y), \quad \tilde{b}(y) = \beta/\cosh(y). \tag{29}$$

Introduction of the test functions $\tilde{a}(y)$ and $\tilde{b}(y)$ into the Lagrangian (24) leads to the following equation for the action integral

$$S = 2\alpha^2\left(\frac{\mu}{3} - 1\right) - \mu\beta^2 + \Gamma\alpha\beta\frac{\pi}{2}, \tag{30}$$

and the dispersion relation $\Gamma^2(\mu)$ is obtained from the conditions $\partial S/\partial\alpha = \partial S/\partial\beta = 0$.

$$\Gamma^2(\mu) = \frac{32}{\pi^2}\mu\left(1 - \frac{\mu}{3}\right). \tag{31}$$

Finally, replacing the parameter μ defined with Eq. (13) we have a detailed structure of the instability growth rate for array soliton solution of the CDNLS equation

$$\Gamma = \frac{8\sqrt{2}\,\sin(s/2)\sqrt{2\lambda^2 - 4\sin^2(s/2)}}{\sqrt{3}\pi\lambda^2} \tag{32}$$

C. Numerical solution

The eigenvalue Eq. (22) are solved numerically. The parity of the solution (even) and the restriction on the square integrable perturbations enable us to look for the solution of the system (22) in the interval $[0, \infty)$ with the boundary conditions

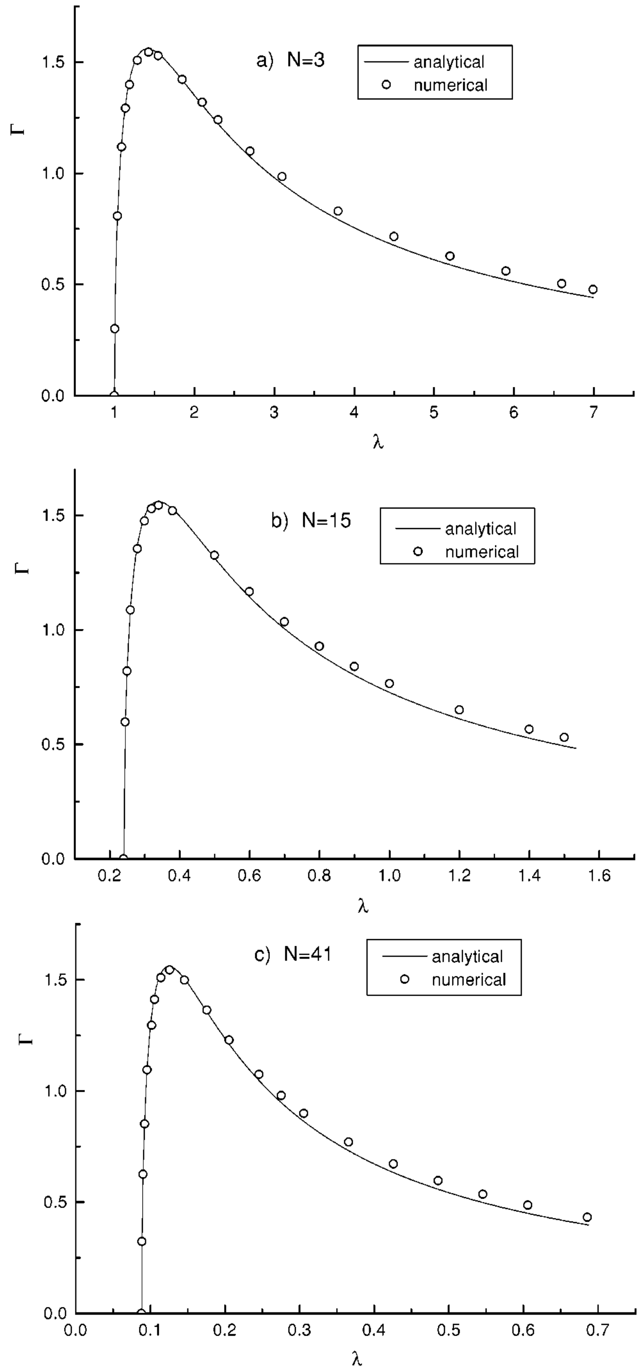

FIG. 1. Comparison of analytically and numerically calculated normalized growth rates Γ as a function of the soliton amplitude λ for different number of array elements.

$$\left.\frac{da(y)}{dy}\right|_{y=0} = \left.\frac{db(y)}{dy}\right|_{y=0} = 0,$$

$$a(\infty) = b(\infty) = 0. \tag{33}$$

The set of eigenvalues and corresponding eigenfunctions satisfying the above boundary conditions are calculated with the "shooting method" [20] adapted here for our problem. Numerically calculated growth rates $\Gamma(\lambda)$ together with the

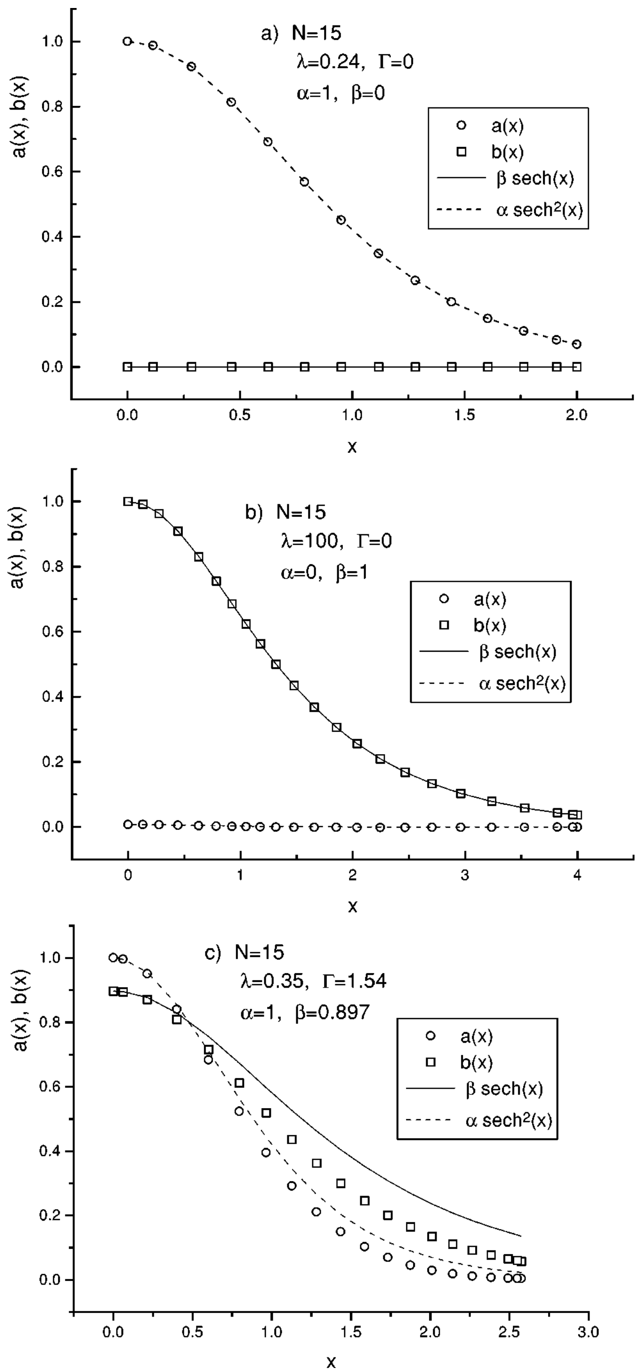

FIG. 2. Eigenfunctions for the array with 15 elements for different values of λ: (a) $\lambda = 0.24$ ($\Gamma \approx 0$), near the first marginally stable state: (b) $\lambda = 100$ ($\Gamma \approx 0$), near the second marginally stable state; (c) $\lambda = 0.35$ ($\Gamma \approx 1.54$), between two marginally stable states.

results from the analytical solution (32) for arrays with three different number of elements ($N = 3;15;41$), are ploted in Fig. 1. A good agreement between the results is obvious indicating that our choice of the test functions for the variational approach was successful. As an illustration we plot in Fig. 2, the test functions (29) together with the numerically calculated eigenfunctions for $N = 15$. Results show a better agreement near the marginally stable states then inside the instability region.

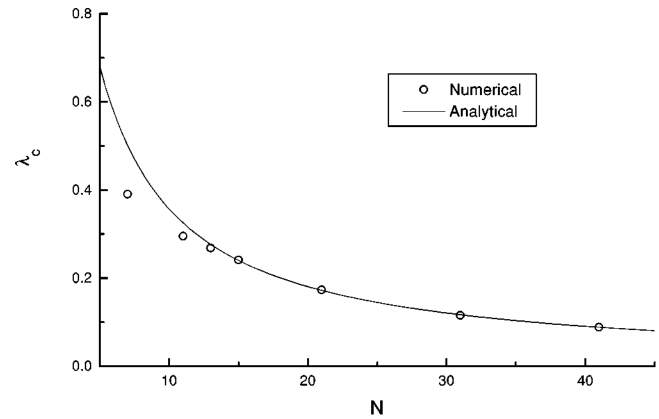

FIG. 3. Analytically and numerically calculated instability threshold values.

The whole previous analysis is devoted to the case of circularly arranged array using periodic boundary conditions. However, a question of possibilities to apply some of these results to the case of open (linearly arranged) arrays naturally arises. Analytical estimates based on the continuum approximation for the threshold of initiation the quasicollapse process in linearly arranged NFA with a Gaussian initial pulse in both dimensions, continuous and discrete, is given in Ref. [7]. Here, we will try to find a threshold for the initial pulse in a form of uniformly distributed solitons (6) along the array elements. It corresponds to the situation where an circular array with a stationary pulse in a form of array solitons is opened between two arbitrary array elements. In this situation the array soliton solution of the circular array configuration no longer satisfy CDNLS equation with a new boundary configuration (open array). To check the pulse dynamics in this configuration we have performed numerical simulations of CDNLS equation based on the split-step Fourier method. Numerical results show existence of the threshold of initiation quasicollapse process that coincides with the instability condition (21) for the case of circular array. The matching of the thresholds is practically exact when the number of array elements exceeds 15, as shown in Fig. 3.

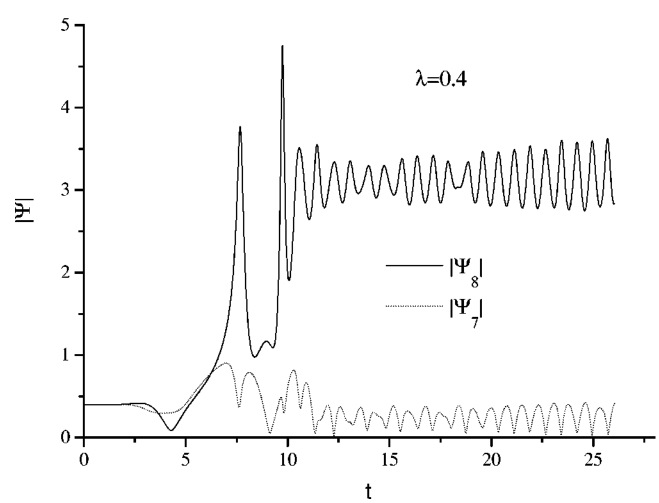

FIG. 4. Evolution of the maximal amplitude in the central (ψ_8) and its neighbor (ψ_7) element.

The initially uniformly distributed (along the array elements) pulse energy during the quasicollapse process is concentrating in the middle elements of the array evolving toward the stationary state in a form of multidimensional soliton [9]. An illustration of this dynamics for an open array with 15 elements is shown in Fig. 4.

IV. CONCLUSION

In this work we have studied in detail the problem of the stability of the soliton array solution of CDNLS equation with periodic boundary conditions (circular array). By virtue of the variational approach originally used for the corresponding continuous problems we have recovered the results for the instability threshold [11,13] and obtained a detail structure of the instability growth rate. Analytical results have been checked numerically and a good agreement has been found.

In addition we have proved numerically that the results for the instability threshold for the circular array is applicable to obtain a quasicollapse threshold for the case of open arrays with the initial pulses in a form of array solitons.

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