Modulation instability in two-dimensional nonlinear Schrödinger lattice models with dispersion and long-range interactions

Ljupčo Hadžievski, Milutin Stepić, and Miloš M. Škorić
Vinča Institute of Nuclear Sciences, P.O.B. 522, 11001 Belgrade, Serbia and Montenegro
(Received 19 December 2002; revised manuscript received 20 February 2003; published 30 July 2003)

The problem of modulation instability of continuous wave and array soliton solutions within the framework of a two-dimensional continuum-discrete nonlinear Schrödinger lattice model, which accounts for dispersion and long-range interactions between elements, is investigated. The linear stability analysis based on an energy principle and a variational approach, which were originally developed for the continuum nonlinear Schrödinger model, is proposed. Regions of instability are identified and analytical expressions for the corresponding thresholds and the growth rate spectra are calculated.

DOI: 10.1103/PhysRevB.68.014305 PACS number(s): 42.81.Dp, 42.65.Tg

I. INTRODUCTION

Mathematical models describing dynamical properties of the systems with interplay between nonlinearity, discreteness, and dispersion attract a growing interest due to their rich applicability in different physical problems. There are many nonlinear physical systems which are both discrete and continuous, such as nonlinear fiber arrays (NFA),[1–13] arrays of coupled Josephson's junctions[14,15] elastic energy transfer in anharmonic crystals,[16,17] DNA molecule chains, etc. Such systems show a complex dynamical behavior exhibiting diverse physical properties like wave instabilities, solitonlike localized structures, quasi-collapse (blowup solutions), pattern formation, and spatiotemporal chaos. Mathematical modeling of these systems often leads to one of the discrete or continuum-discrete variants of the universal nonlinear evolution equations such as nonlinear Schrödinger (NLS), sine-Gordon, Korteweg–de Vries, Klein-Gordon, and Kadomtsev-Petviashvili equations. The simplest and also most extensively studied are NLS models. In the physical situation where the dispersion along the lattice elements can be neglected, the NLS lattice model is described by the discrete NLS (DNLS) equation where dynamical properties of the system are determined by an interplay between nonlinearity and discreteness. Incorporation of the dispersion along the lattice elements into the NLS lattice model leads to a more complicated continuum-discrete NLS (CDNLS) type of equation, where dynamical properties of the system are determined by an interplay between nonlinearity, discreteness, and dispersion. The CDNLS models were intensively studied mainly for one-dimensional (1D) NLS lattices with short-range (nonlocal) interactions by using a nearest-neighbor approximation.[4–9,11–13,18,19] Two-dimensional (2D) lattices based on the CDNLS model were studied in Refs. 4 and 5 and within the DNLS model in Refs. 20–23. However, some physical systems cannot be described in the framework of nearest-neighbor approximation and the effect of long-range interactions between the lattice elements must be taken into account. Examples are DNA molecule chains with long-range Coulomb interactions, excitation transfer in molecular crystals and vibron energy transport in biopolymers with dipole-dipole interactions. The effects of long-range dispersive interactions in the 1D DNLS lattice model was investigated in Refs. 24–26. The long-range interaction model with a power law dependence on the distance between interacting elements was used in Ref. 24. A modified interaction model in a form of Joncqière's function convenient to cover different physical situations from nearest-neighbor interactions to ultra-long-range interactions was discussed in Ref. 25. The dynamics of the 2D DNLS lattice model with long-range dipole-dipole interactions was studied in Ref. 27. To our knowledge, the present paper is a rare attempt to analyze properties of 2D (also 1D) lattices within CDNLS models accounting for dispersion and long-range interactions.

The goal of this work is to investigate an important problem of modulation instability of continuous wave (cw) and array soliton (AS) solutions in the framework of a general continuum-discrete nonlinear Schrödinger (CDNLS) lattice model describing dynamics in a 2D lattice with dispersion and long-range interactions between elements. In Sec. II we define the basic evolution equation and give the continuous wave (cw) and array soliton solutions of the model. In Sec. III we describe a linear stability analysis based on an energy principle and variational approach which were originally developed for the continuum NLS models.[28,29] We obtain explicit analytical expressions for the instability thresholds and the growth rate spectra which also recover results for 1D lattices and the nearest-neighbor interaction model. Finally, we summarize our results in Sec. IV.

II. THE MATHEMATICAL MODEL

The basic mathematical model describing the two-dimensional lattice with nonlocal nonlinear interacting elements in anomalous dispersion regime has a form of continuum-discrete nonlinear Schrödinger equation

$$i\frac{\partial \psi_{\vec{r}}}{\partial t}+\frac{\partial^2 \psi_{\vec{r}}}{\partial z^2}+2\psi_{\vec{r}}|\psi_{\vec{r}}|^2+\sum_{\vec{r}'\,(\vec{r}'\neq\vec{r})} J_{|\vec{r}'-\vec{r}|}(\psi_{\vec{r}'}-\psi_{\vec{r}})=0, \tag{1}$$

where $\vec{r}=(n,m,0)(n=0,\pm 1,\pm 2,\ldots,N;\ m=0,\pm 1,\pm 2,\ldots,M)$ is the discrete lattice vector in a x-y plane, z is the spatial continuous coordinate along the lattice elements, and $\psi_{\vec{r}}=\psi_{n,m}$ is the wave function into the (n,m)th lattice element. The nonlocal interaction term $J_{|\vec{r'}-\vec{r}|}$ describes a long-range isotropic coupling between lattice elements and depends on the distance between interacting elements. This interaction model is quite general and enables a mathematical modeling of a variety of discrete dispersive physical systems with long-range interactions. The well-known interaction model for the 1D DNLS lattice model with a power law dependence on the distance between interacting elements was originally proposed in Ref. 24. In our case, for the CDNLS model (1) with a regularly spaced 2D lattice with interelement distance equal to 1, the power law dependence can be written in the form

$$J_{|\vec{r'}-\vec{r}|}=\frac{1}{|\vec{r'}-\vec{r}|^p}. \tag{2}$$

This interaction model can conveniently describe a wide class of different discrete dispersive physical systems with a long-range isotropic interactions, such as: DNA molecule chains with a long-range Coulomb interaction $(p=1)$, propagation of optical pulses in nonlinear fiber arrays, and excitation transfer in quasi-two-dimensional molecule crystals $(p=3)$. On the other hand, for the sufficiently large exponent p, the model Eq. (1) exhibits the same qualitative features as the CDNLS equation with nearest-neighbor interactions.

The CDNLS equation (1) has a Hamiltonian structure and can be written as

$$i\frac{\partial \psi_{\vec{r}}}{\partial t}=\frac{\delta H}{\delta \psi_{\vec{r}}^*}, \tag{3}$$

where H is the Hamiltonian defined by

$$H=\sum_{\vec{r}}\int_{-\infty}^{\infty}\left(\sum_{\vec{r'}\ (\vec{r'}\neq\vec{r})} J_{\vec{r'}-\vec{r}}(\psi_{\vec{r'}}-\psi_{\vec{r}})\psi_{\vec{r}}^* + |(\psi_{\vec{r}})_z|^2-|\psi_{\vec{r}}|^4\right)dz, \tag{4}$$

where index z stands for the partial derivative with respect to variable z.

The number of quanta P (L^2 norm) is another conserved quantity of the Eq. (1)

$$P=\sum_{\vec{r}}\int_{-\infty}^{\infty}|\psi_{\vec{r}}|^2 dz. \tag{5}$$

For the lattice with periodic boundary conditions imposed on the discrete dimensions $\vec{r}$ we can consider a set of lattice independent stationary solutions of Eq. (1) in the form

$$\psi_{\vec{r}}=f(z)e^{i\lambda^2 t}, \tag{6}$$

where λ is a real parameter. We shall restrict our stability study to two particularly simple and most frequently studied stationary solutions: the first one is a uniform, continuous wave (cw) solution $f_{cw}=\lambda/\sqrt{2}$, while the second one is an array soliton (AS), given by $f_{as}=\lambda/\cosh(\lambda z)$. In both cases the parameter λ is related to the amplitude of the wave function.

III. STABILITY ANALYSIS

In order to study the stability property of stationary solutions (6) we introduce small modulations, in a form of square integrable, perturbations

$$\psi_{\vec{r}}(x,t)=[f(z)+\delta f_{\vec{r}}(z,t)]e^{i\lambda^2 t}, \quad |\delta f_{\vec{r}}|\ll|f|. \tag{7}$$

Substituting Eq. (7) into Eq. (1) and linearizing with respect to small perturbations $\delta f_{\vec{r}}$, we arrive at

$$i\frac{\partial \delta f_{\vec{r}}}{\partial t}+\frac{\partial^2 \delta f_{\vec{r}}}{\partial z^2}-\lambda^2\delta f_{\vec{r}}+4|f|^2\delta f_{\vec{r}}+2|f|^2\delta f_{\vec{r}}^* + \sum_{\vec{r'}\ (\vec{r'}\neq\vec{r})} J_{\vec{r'}-\vec{r}}(\delta f_{\vec{r'}}-\delta f_{\vec{r}})=0. \tag{8}$$

To find sufficient conditions for the linear instability we will assume perturbations with a simple harmonic dependence on the discrete dimensions $\vec{r}$ in a form[5]

$$\delta f_{\vec{r}}(z,t)=(a+ib)\cos(k_n n)\cos(k_m m), \tag{9}$$

where $k_n=2\pi/(2N+1)$ and $k_m=2\pi/(2M+1)$ are discrete wave numbers. From Eq. (8) with perturbations (9), the following eigenvalue problem is obtained:

$$\frac{\partial b(z,t)}{\partial t}=-\hat{L}_+ a(z,t),$$
$$\frac{\partial a(z,t)}{\partial t}=\hat{L}_- b(z,t). \tag{10}$$

The linear second-order differential operators $\hat{L}_\pm$ are defined by

$$\hat{L}_+=-\frac{\partial^2}{\partial z^2}+\lambda^2-6f^2(z)+4\,\Sigma\,(N,M),$$
$$\hat{L}_-=-\frac{\partial^2}{\partial z^2}+\lambda^2-2f^2(z)+4\,\Sigma\,(N,M). \tag{11}$$

The complete discrete properties of the system described by the operators $\hat{L}_\pm$ are taken into account through the interaction term $\Sigma\,(N,M)$. It is straightforward to obtain the expression for $\Sigma\,(N,M)$ from the nonlocal interaction's (last) term in Eq (8), after making the substitution of the perturbation, given by the explicit formula (9)

$$\Sigma(N,M)=\sum_{n=1}^{N} J_{n,0}\sin^2\left(\frac{k_n n}{2}\right)+\sum_{m=1}^{M} J_{0,m}\sin^2\left(\frac{k_m m}{2}\right)-\sum_{n=1}^{N}\sum_{m=1}^{M} J_{n,m}[\cos(k_n n)\cos(k_m m)-1]. \tag{12}$$

The interaction term (12) depends on the lattice dimensions and the form of interaction between lattice elements.

A. Stability of the cw solution

For the case of cw solution ($f_{cw}=\lambda/\sqrt{2}$) the differential operators (11) are homogeneous and stability analysis is straightforward. The Fourier transform ($e^{-i\omega t+ikz}$) of Eqs. (10) gives the following dispersion relation:

$$\omega^2=(k^2+4\Sigma)(k^2-2\lambda^2+4\Sigma). \tag{13}$$

The instability occurs for $\omega^2<0$, which leads to the following instability threshold:

$$\lambda^2>\frac{k^2}{2}+2\Sigma(N,M). \tag{14}$$

As seen from Eq. (14), the lowest threshold is for an excitation of small wave-number (long-wavelength) perturbations corresponding to the modulation instability. Dispersion relation (13) and the instability threshold (14) according to the dimensionality of the lattice (one-dimensional or two-dimensional) and type of the interaction (nearest-neighbor or long-range interactions) is represented by four explicit analytical expressions:

(a) One-dimensional lattice $\vec{r}=(n,0,0)$ *with nearest-neighbor interactions:*

$$\omega^2=\left[k^2+4\sin^2\left(\frac{k_n}{2}\right)\right]\left[k^2+4\sin^2\left(\frac{k_n}{2}\right)-2\lambda^2\right], \tag{15}$$

$$\lambda^2>\frac{k^2}{2}+2\sin^2\left(\frac{\pi}{2N+1}\right). \tag{16}$$

Results given by relations (15) and (16) coincide to the corresponding ones obtained in Refs. 5 and 9.

(b) One-dimensional lattice $\vec{r}=(n,0,0)$ *with long-range interactions:* The instability threshold in this case reads

$$\lambda^2>\frac{k^2}{2}+2\sum_{n=1}^{N} J_n\sin^2\left(\frac{\pi}{2N+1}n\right). \tag{17}$$

For the interaction model with a power law dependence on the distance between interacting elements (2), such as $J_n=1/n^p$, the instability threshold (17) reads

$$\lambda^2>\frac{k^2}{2}+2\sum_{n=1}^{N}\frac{\sin^2\left(\frac{\pi}{2N+1}n\right)}{n^p}. \tag{18}$$

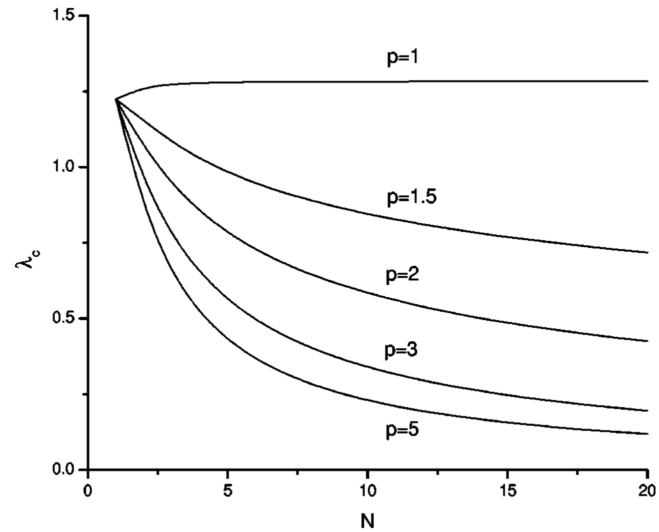

FIG. 1. Dependence of the instability threshold λ_c on the size of the one-dimensional lattice *N*, with long-range interactions for different values of *p*.

The instability threshold λ_c for long-range interactions as a function of the size of the 1D lattice (N) for different values of p is plotted in Fig. 1. The curve for $p=5$ practically corresponds to the result (16) for the nearest-neighbor interactions model. The above results show that due to the increased inertia of the system the instability threshold for the long-range interactions is higher than the corresponding threshold for the nearest-neighbor interactions.

(c) The two-dimensional lattice $\vec{r}=(n,m,0)$ *with nearest-neighbor interactions:* The instability threshold for this case reads

$$\lambda^2>\frac{k^2}{2}+2\left[1-\cos\left(\frac{k_n-k_m}{2}\right)\cos\left(\frac{k_n+k_m}{2}\right)\right]. \tag{19}$$

For highly elongated 2D lattices, with $N\gg M$, perturbations dominantly develop along the longer dimension of the lattice and the instability threshold (19) approaches values for the instability threshold for 1D lattices (16).

(d) The two-dimensional lattice $\vec{r}=(n,m,0)$ *with long-range interactions:*

$$\lambda^2>\frac{k^2}{2}+2\left\{\sum_{n=1}^{N} J_{n,0}\sin^2\left(\frac{k_n n}{2}\right)+\sum_{m=1}^{M} J_{0,m}\sin^2\left(\frac{k_m m}{2}\right)-\sum_{n=1}^{N}\sum_{m=1}^{M} J_{n,m}[\cos(k_n n)\cos(k_m m)-1]\right\}. \tag{20}$$

For the interaction model with a power law dependence on the distance between the interacting elements (2), $J_{n,m}=1/(n^2+m^2)^{p/2}$, the instability threshold (20) becomes

$$\lambda^2>\frac{k^2}{2}+2\left\{\sum_{n=1}^{N}\frac{\sin^2(k_n n/2)}{n^p}+\sum_{m=1}^{M}\frac{\sin^2(k_m m/2)}{m^p}-\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{[\cos(k_n n)\cos(k_m m)-1]}{(n^2+m^2)^{p/2}}\right\}. \tag{21}$$

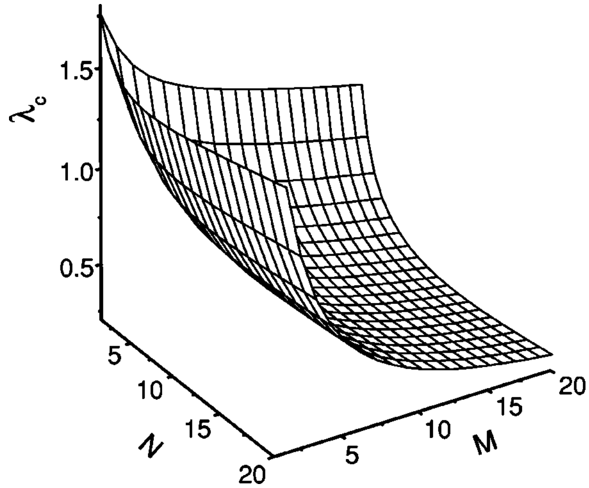

FIG. 2. The instability threshold λ_c as a function of the size of the two-dimensional lattice (N,M) with long-range interactions for $p=3$.

The instability threshold λ_c for long-range interactions as a function of the size of the 2D lattice (N,M) for $p=3$ is presented in Fig. 2. The value $p=3$ corresponds to the case of isotropic dipole-dipole interactions discussed in Ref. 27 for two-dimensional DNLS lattice model. The instability threshold decreases with the size of the lattice and has a minimum for the square lattice $N=M$. Increasing p leads to a decrease of λ_c and for large values of p approaches the result given by Eq. (19) for the nearest-neighbor interactions model. As seen on inspection, the dispersion term in Eq. (1) increases the cw instability threshold (more stable); the effect which vanishes for the long-wavelength perturbation.

B. Stability of array soliton solutions

Stability analysis of AS solutions becomes more complicated due to an explicit z dependence of the differential operators $\hat{L}_\pm$. However, the fact that the discrete properties of the system are incorporated into the operators $\hat{L}_\pm$ only via the $\Sigma(N,M)$ term, enables a direct application of the mathematical methods developed for stability analysis of continuum models. To calculate the instability threshold and detailed spectra of the growth rate, we generalize and apply to our 2D CDNLS model[13] an energy principle by Laedke and Spatschek[28] and a variational method by Rypdal and Rasmussen,[29] which were originally introduced for stability studies of the continuum NLS equation solutions.

The energy principle is applicable when the linearized evolution equations with respect to perturbations (10) can be written in a standard form $\hat{L}_-^{-1}a_{tt}=-\hat{L}_+a$ with $\hat{L}_-^{-1}$ positive definite. For this case the positive definiteness of the operator $\hat{L}_+$ is a necessary and sufficient condition for the Ljapunov stability of the system. The detailed proof is given in Refs. 28 and 29. However, for CDNLS, in the derivation of Eq. (10) we have assumed perturbations (9) with a simple harmonic dependence on the discrete variables (n,m). Strictly, while for continuum models harmonic perturbation ansatz provides a complete basis (Fourier space) for the general perturbations; this is not a case for discrete models. It basically means that in our case the applied energy principle will give necessary and sufficient conditions for the stability under the assumed type of discrete perturbations (9).

For our further calculations it is convenient to substitute $\lambda z\to z$ and to express operators $\hat{L}_\pm$ in the form

$$\hat{L}_+=\lambda^2(\hat{S}_+ +\mu-5),$$

$$\hat{L}_-=\lambda^2(\hat{S}_- +\mu-1), \tag{22}$$

where μ is the parameter containing information about the discreteness of the system, defined by

$$\mu=\frac{4\,\Sigma}{\lambda^2}, \tag{23}$$

and $\hat{S}_\pm$ are Sturm-Liouville-type operators

$$\hat{S}_+=-\frac{\partial^2}{\partial z^2}+6\tanh^2(z),$$

$$\hat{S}_-=-\frac{\partial^2}{\partial z^2}+2\tanh^2(z). \tag{24}$$

These operators possess a well-known spectra.[30] The smallest eigenvalues $\sigma_\pm^{(0)}$ and corresponding eigenfunctions $\psi_\pm^{(0)}$ in the discrete part of the spectrum are

$$\sigma_-^{(0)}=1;\quad \psi_-^{(0)}=1/\cosh(z),$$

$$\sigma_+^{(0)}=2;\quad \psi_+^{(0)}=1/\cosh^2(z). \tag{25}$$

The procedure of the energy principle described in Refs. 28 and 29 demands one to find regions of the parameter μ where operators $\hat{L}_\pm$ are positive definite or indefinite. Since the Sturm-Liouville operators $\hat{S}_\pm$ are positive definite and possess only positive eigenvalues, it is straightforward to find that the operator $\hat{L}_-$ is positive definite for $\mu>0$, while the operator $\hat{L}_+$ is indefinite for $0<\mu<3$ and positive definite for $\mu>3$. According to the energy principle,[29] the sufficient conditions for the instability are satisfied in the region $0<\mu<3$, where the operator $\hat{L}_+$ is indefinite. The system is stable under the assumed perturbations (9) for $\mu>3$, where operators $\hat{L}_\pm$ are positive definite. These results lead to the next instability condition

$$\lambda>\lambda_c=\frac{2\sqrt{\Sigma(N,M)}}{\sqrt{3}}. \tag{26}$$

If we compare the above instability threshold for AS solutions with the threshold for CW solutions given by Eq. (14) it becomes obvious that for $k=0$ the difference comes only within a numerical factor $\sqrt{2/3}\approx 0.8165$. It means that all corresponding explicit results for the instability thresholds for AS solutions can be derived from the expressions given by Eqs. (14)–(21) for the instability thresholds of the cw solutions; by taking $k=0$ and just multiplying by a factor 0.8165. It also means that the shapes of the curves displayed in Figs. 1 and 2 are the same as in the case of AS solutions. For the 1D lattice with nearest-neighbor interactions, Eq. (26) readily recovers earlier results obtained in Refs. 5, 9, and 13.

The application of the above energy principle to the stability of AS solutions proves the existence of exponentially growing modes and gives the threshold value (26), without providing any further insight. Therefore, in order to calculate the growth rate spectral structure of the instability, we apply a variational approach, originally introduced for the continuum NLS equation in Ref. 29 and first generalized to a continuum-discrete 1D NLS equation with nearest-neighbor interactions, by these authors.[13] For the normal exponentially growing modes $a(t,z)=a(z)\exp(\gamma t)$; $b(t,z)=b(z)\exp(\gamma t)$ with the growth rate γ, the eigenvalue equations (10) after the substitution $\lambda z \rightarrow z$ are transformed into

$$\hat{L}_{+}a(z)=-\Gamma b(z),$$
$$\hat{L}_{-}b(z)=\Gamma a(z), \tag{27}$$

where $\Gamma=\gamma/\lambda^2$ is the normalized growth rate. The above equations can be derived from the variation of the action

$$\delta S=\delta\int_{-\infty}^{\infty}\mathcal{L}(a,a_z,b,b_z,z)dz, \tag{28}$$

where the Lagrangian $\mathcal{L}$ is given by

$$\mathcal{L}=\frac{1}{2}(a_z^2+b_z^2)+\left[\frac{\mu+1}{2}-\frac{3}{\cosh^2(z)}\right]a^2+\left[-\frac{\mu+1}{2}+\frac{1}{\cosh^2(z)}\right]b^2+\Gamma ab. \tag{29}$$

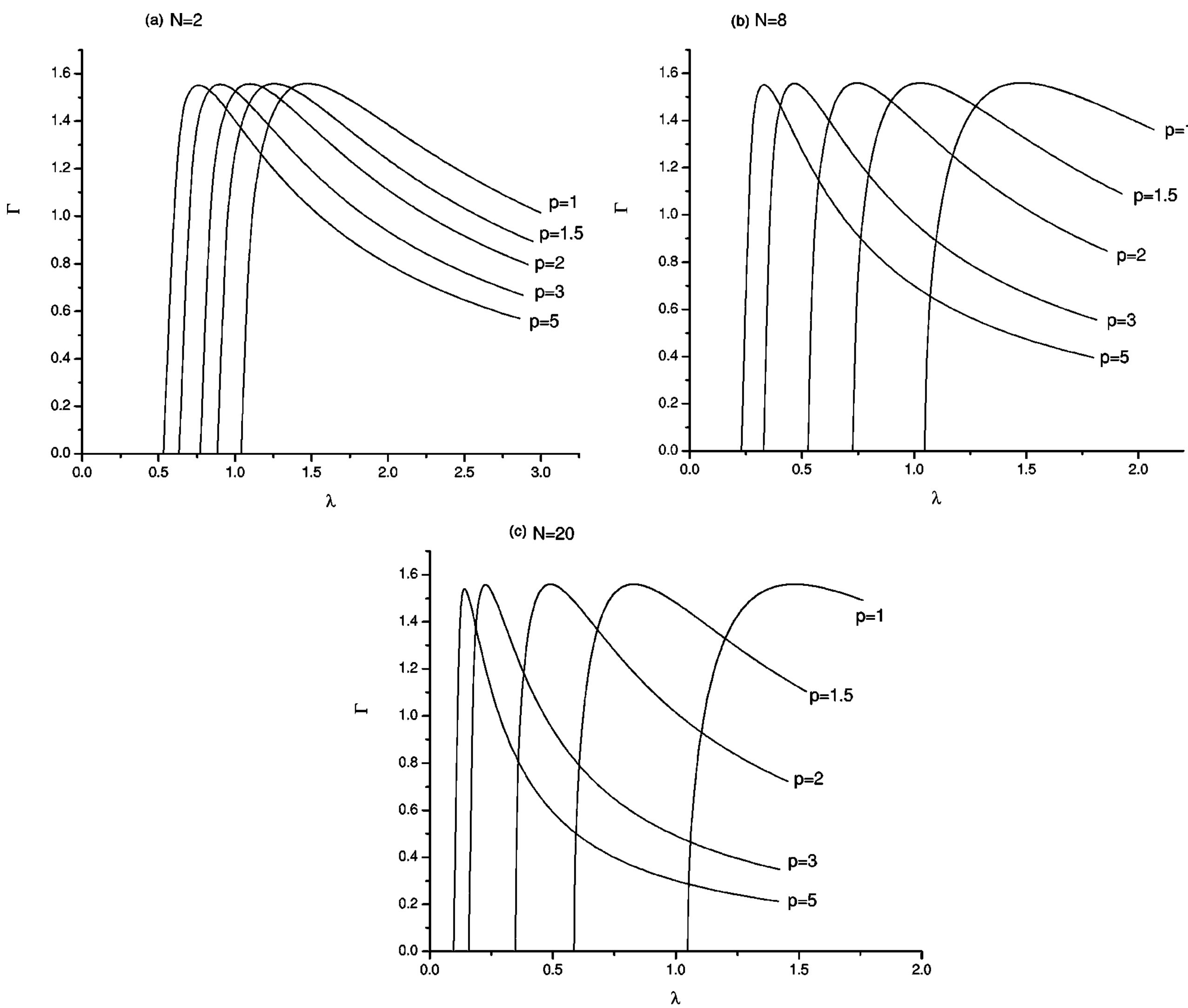

FIG. 3. Dependence of the growth rates Γ on the soliton amplitude λ for three different one-dimensional lattices with long-range interactions: (a) $N=2$, (b) $N=8$, and (c) $N=20$.

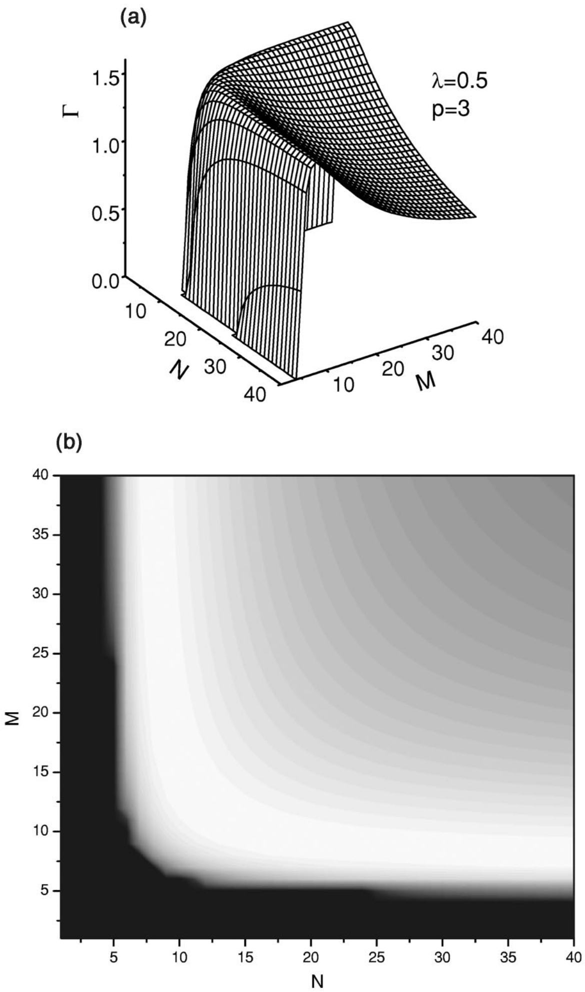

FIG. 4. Growth rates Γ of the instability of the array soliton solution with amplitude $\lambda = 0.5$, as a function of the size of the two-dimensional lattice (N,M), with long-range interactions for $p=3$: (a) surface $\Gamma(N,M)$; (b) corresponding gray scale map of the projection on the N-M plane.

The basic idea of the variational approach is to define a set of test functions $\tilde{a}(z)$ and $\tilde{b}(z)$ with some variational parameters and to calculate the action integral S. It is obvious that with this approach, obtained results will critically depend on our choice of the test functions. It was shown and also numerically confirmed in Refs. 13 and 29 that a good choice for the test functions are eigenfunctions of Eqs. (27) for the marginally stable states $(\Gamma=0)$.

$$a(z)=0, \quad b(z)=\frac{1}{\cosh(z)}, \quad \mu=0$$

$$a(z)=\frac{1}{\cosh^2(z)}, \quad b(z)=0, \quad \mu=3. \tag{30}$$

Assuming test functions with two variational parameters α and β in the form

$$\tilde{a}(z)=\frac{\alpha}{\cosh^2(z)}, \quad \tilde{b}(z)=\frac{\beta}{\cosh(z)}, \tag{31}$$

we calculate the action integral

$$S=2\alpha^2\left(\frac{\mu}{3}-1\right)-\mu\beta^2+\frac{\pi}{2}\Gamma\alpha\beta. \tag{32}$$

The following expression for the growth rate structure

$$\Gamma^2(\mu)=\frac{32}{\pi^2}\mu\left(1-\frac{\mu}{3}\right), \tag{33}$$

is obtained after variation of Eq. (32), from the conditions $\partial S/\partial\alpha=\partial S/\partial\beta=0$.

The instability threshold λ_c corresponds to the marginally stable mode $\Gamma=0$ in the dispersion relation (33). The expression for the threshold calculated from Eq. (33) coincides, as expected, with the expression (26) obtained by applying the energy principle.

The dispersion relation (33) has the same structure as results given in Ref. 29 for the continuum NLS equation and in Ref. 13 for the 1D CDNLS equation with nearest-neighbor interactions model, because complete discrete properties of the system are incorporated only via the parameter μ, which is defined by Eq. (23). Replacing the particular expressions for μ into Eq. (33) for lattices with different dimensionality (one-dimensional or two-dimensional) and type of the interactions (nearest-neighbor or long-range interactions) we can readily obtain explicit formulas for the corresponding growth rate structure.

(a) One-dimensional lattice $\vec{r}=(n,0,0)$ *with nearest-neighbor interactions:*

$$\Gamma=\frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sin\left(\frac{\pi}{2N+1}\right)\sqrt{3\lambda^2-4\sin^2\left(\frac{\pi}{2N+1}\right)}. \tag{34}$$

(b) One-dimensional lattice $\vec{r}=(n,0,0)$ *with long-range interactions:*

$$\Gamma=\frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{\sum_{n=1}^{N} J_n\sin^2\left(\frac{\pi}{2N+1}n\right)}\times\sqrt{3\lambda^2-4\sum_{n=1}^{N} J_n\sin^2\left(\frac{\pi}{2N+1}n\right)}. \tag{35}$$

For the interaction model with a power law dependence on the distance between interacting elements (2), $J_n=1/n^p$ the growth rate (35) reads

$$\Gamma=\frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{\sum_{n=1}^{N}\frac{\sin^2\left(\frac{\pi}{2N+1}n\right)}{n^p}}\times\sqrt{3\lambda^2-4\sum_{n=1}^{N}\frac{\sin^2\left(\frac{\pi}{2N+1}n\right)}{n^p}}. \tag{36}$$

Figures 3(a)–(c) show the dependence of the growth rate Γ on the soliton amplitude, for three different 1D lattices: (a) $N=2$, (b) $N=8$, and (c) $N=20$. The curves for large p practically correspond to the results for the nearest-neighbor interactions model.[13] The growth rate is less sensitive to the variation of p for the lattices with lower number of elements and for $N=1$ (one-dimensional lattice with three elements) all curves degenerate into a single one which coincides with the growth rate for the nearest-neighbor interactions model.

(c) Two-dimensional lattice $\vec{r}=(n,m,0)$ with nearest-neighbor interactions:

$$\Gamma=\frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{1-\cos\left(\pi\frac{k_n-k_m}{2}\right)\cos\left(\pi\frac{k_n+k_m}{2}\right)}\sqrt{3\lambda^2-4+4\cos\left(\pi\frac{k_m-k_n}{2}\right)\cos\left(\pi\frac{k_n+k_m}{2}\right)}. \tag{37}$$

(d) Two-dimensional lattice $\vec{r}=(n,m,0)$ with long-range interactions:

$$\Gamma=\frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{\sum_{n=1}^{N}J_{n,0}\sin^2(k_n n/2)+\sum_{m=1}^{M}J_{0,m}\sin^2(k_m m/2)-\sum_{n=1}^{N}\sum_{m=1}^{M}J_{n,m}[\cos(k_n n)\cos(k_m m)-1]}$$
$$\times\sqrt{3\lambda^2-4\left\{\sum_{n=1}^{N}J_{n,0}\sin^2(k_n n/2)+\sum_{m=1}^{M}J_{0,m}\sin^2(k_m m/2)-\sum_{n=1}^{N}\sum_{m=1}^{M}J_{n,m}[\cos(k_n n)\cos(k_m m)-1]\right\}}. \tag{38}$$

For the interaction model with a power law dependence on the distance (2) $J_{n,m}=1/(n^2+m^2)^{p/2}$, the growth rate (38) reads

$$\Gamma=\frac{8\sqrt{2}}{\lambda^2\pi\sqrt{3}}\sqrt{\sum_{n=1}^{N}\frac{\sin^2(k_n n/2)}{n^p}+\sum_{m=1}^{M}\frac{\sin^2(k_m m/2)}{m^p}-\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{\cos(k_n n)\cos(k_m m)-1}{(n^2+m^2)^{p/2}}}$$
$$\times\sqrt{3\lambda^2-4\left[\sum_{n=1}^{N}\frac{\sin^2(k_n n/2)}{n^p}+\sum_{m=1}^{M}\frac{\sin^2(k_m m/2)}{m^p}-\sum_{n=1}^{N}\sum_{m=1}^{M}\frac{\cos(k_n n)\cos(k_m m)-1}{(n^2+m^2)^{p/2}}\right]}. \tag{39}$$

In Figs. 4(a) and 4(b) we show the growth rate Γ for the instability of the array soliton with the amplitude $\lambda=0.5$ as a function of the size of the two-dimensional lattice (N,M) with long-range interactions for $p=3$. Figure 4(a) represents the surface $\Gamma(N,M)$, while Fig. 4(b) is the corresponding gray scale map of the projection on the N–M plane. The black area in Fig. 4(b) is the region below the instability threshold.

IV. CONCLUSION

In this work, we have analytically studied detailed stability properties of the continuous wave and array soliton stationary solutions in the framework of the general model of two-dimensional nonlinear lattices with dispersion and long-range interactions, described by the 2D CDNLS equation. The linear stability of the array soliton solution is solved by applying the energy principle and the variational method which were originally developed for the continuum NLS equation.[28,29] We have found above solutions to be unstable and have calculated explicit expressions for the instability thresholds and growth rate spectra valid for the 2D CDNLS lattice with a long-range isotropic coupling between lattice elements. Explicit formulas for the long-range isotropic interactions with a power law dependence on the distance between interacting elements are also derived. Our results for highly elongated 2D lattices with large p recover the expression for the one-dimensional lattice with nearest-neighbor interactions, obtained in earlier papers.[4,5,9,13,18] This study seems to be the first attempt to address the stationary solution stability in 2D lattices within the CDNLS model with dispersion and long-range interactions. Therefore, our findings can be particularly relevant to different physical systems modelled by 2D DNLS and CDNLS lattices models with long-range interactions, e.g., nonlinear optical fiber arrays,[1–13] DNA molecule chains,[31] molecular crystal excitation,[32] and vibron transport in biopolimers,[33] etc.

The results presented in our study are based on the linear stability analysis and indicate a presence of small exponentially growing modes in the system, giving no predictions on the subsequent nonlinear evolution stage. Based on the results for 1D and 2D CDNLS lattice models with nearest-neighbor interactions[4,5,13] and for 2D DNLS lattice models (without dispersion) with long-range interactions,[21–23,27] it is plausible to expect a nonlinear development of the quasicollapse process[4,5] and solitary structures localized in both continuum and discrete dimensions (continuum-discrete solitary waves). A detail study of these problems in the framework of the 2D CDNLS lattice model with long-range interactions as well as the problem of stability of multidimensional continuum-discrete solitary waves is planned to be given in a separate publication.

ACKNOWLEDGMENTS

The Ministry of Science, Technologies and Development of the Republic of Serbia (Project 1964) has supported this work. One of us (Lj.H.) gratefully acknowledges the hospitality of Aston University, Birmingham (UK) and helpful discussions with the members of the Photonix group.

[1] D. N. Christodoulides and R. I. Joseph, Opt. Lett. **13**, 794 (1988).
[2] H. S. Eisenberg, Y. Silberberg, R. Morandotti, A. R. Boyd, and J. S. Aitchison, Phys. Rev. Lett. **81**, 3383 (1998).
[3] R. Morandotti, U. Peschel, J. S. Aitchison, H. S. Eisenberg, and Y. Silberberg, Phys. Rev. Lett. **83**, 2726 (1999).
[4] A. B. Aceves, G. G. Luther, C. De Angelis, A. M. Rubenchik, and S. K. Turitsyn, Phys. Rev. Lett. **75**, 73 (1995).
[5] A. B. Aceves, C. De Angelis, G. G. Luther, A. M. Rubenchik, and S. K. Turitsyn, Physica D **87**, 262 (1995).
[6] J. M. Soto-Crespo and E. M. Wright, J. Appl. Phys. **70**, 7240 (1991).
[7] P. E. Langridge and W. J. Firth, Opt. Quantum Electron. **24**, S1315 (1992).
[8] A. B. Aceves, C. De Angelis, A. M. Rubenchik, and S. K. Turitsyn, Opt. Lett. **19**, 329 (1994).
[9] A. B. Aceves, C. De Angelis, G. G. Luther, and A. M. Rubenchik, Opt. Lett. **19**, 1186 (1994).
[10] M. I. Weinstein and B. Yeary, Phys. Lett. A **222**, 157 (1996).
[11] S. Darmanyan, I. Relke, and F. Lederer, Phys. Rev. E **55**, 7662 (1997).
[12] I. Relke, Phys. Rev. E **57**, 6105 (1998).
[13] M. Stepić, Lj. Hadžievski, and M. Škorić, Phys. Rev. E **65**, 026604 (2002).
[14] L. M. Floría, J. L. Marín, P. J. Martínez, F. Falo, and S. Aubry, Europhys. Lett. **36**, 539 (1996).
[15] E. Trías, J. J. Mazo, and T. P. Orlando, Phys. Rev. Lett. **84**, 741 (2000).
[16] S. A. Kiselev, S. R. Bickham, and A. J. Sievers, Comments Condens. Matter Phys. **17**, 135 (1995).
[17] S. A. Kiselev and A. J. Sievers, Phys. Rev. B **55**, 5755 (1997).
[18] M. I. Weinstein, Nonlinearity **12**, 673 (1999).
[19] F. T. Hioe, Phys. Rev. Lett. **82**, 1152 (1999).
[20] J. Pouget, M. Remoissenet, and J. M. Tamga, Phys. Rev. B **47**, 14 866 (1993).
[21] E. W. Laedke, K. H. Spatschek, V. K. Meyentsev, S. L. Musher, I. V. Ryzhenkova, and S. K. Turitsyn, Pisma. Zh. Éksp. Teor. Fiz **62**, 652 (1995) [JETP Lett. **62**, 677 (1995)].
[22] P. L. Christiansen, Yu. B. Gaididei, K. Ø. Rasmussen, V. K. Mezentsev, and J. J. Rasmussen, Phys. Rev. B **54**, 900 (1996).
[23] S. Flach, K. Kladko and R. S. MacKay, Phys. Rev. Lett. **78**, 1207 (1997).
[24] Yu. B. Gaididei, S. F. Mingaleev, P. L. Christiansen, and K. Ø. Rasmussen, Phys. Rev. E **55**, 6141 (1997).
[25] P. L. Christiansen, Y. B. Gaididei, F. G. Mertens, and S. F. Mingaleev, Eur. Phys. J. B **19**, 545 (2001).
[26] Yu. B. Gaididei, S. F. Mingaleev, and P. L. Christiansen, Phys. Rev. E **62**, R53 (2000).
[27] P. L. Christiansen, Yu. B. Gaididei, M. Johansson, K. Ø. Rasmussen, V. K. Mezentsev and J. J. Rasmussen, Phys. Rev. B **57**, 11 303 (1998).
[28] E. W. Laedke and K. H. Spatschek, Phys. Rev. Lett. **41**, 1798 (1978).
[29] K. Rypdal and J. J. Rasmussen, Phys. Scr. **40**, 192 (1989).
[30] P. M. Morse and H. Feshback, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953), Chaps. 6 and 12.
[31] A.S. Davydov, *Theory of Molecular Excitons* (Plenum, New York, 1973).
[32] M. Peyrard and A.R. Bishop, Phys. Rev. Lett. **62**, 2755 (1989).
[33] A.C. Scott, Phys. Rep. **217**, 1 (1992).