Dynamics of dark breathers in lattices with saturable nonlinearity

Ljupčo Hadžievski,[1*] **Aleksandra Maluckov,**[2] **and Milutin Stepić**[3]

[1] *Vinča Institute of Nuclear Sciences, P. O. Box 522, 11001 Belgrade, Serbia*
[2] *Faculty of Sciences and Mathematics, University of Niš, P. O. Box 224, 18001 Niš, Serbia*
[3] *Dept. of Optical Technologies, National Metrology Institute, 38116 Braunschweig, Germany*
[*] *Corresponding author: ljupcoh@vin.bg.ac.yu*

Abstract: The problems of the existence, stability, and transversal motion of the discrete dark localized modes in the lattices with saturable nonlinearity are investigated analytically and numerically. The stability analysis shows existence of regions of the parametric space with eigenvalue spectrum branches with non-zeroth real part, which indicates possibility for the propagation of stable on-site and inter-site dark localized modes. The analysis based on the conserved system quantities reveals the existence of regions with a vanishing Peierls-Nabarro barrier which allows transverse motion of the dark breathers. Propagation of the stable on-site and inter-site dark breathers and their free transversal motion are observed numerically.

OCIS codes: (190.0190) Nonlinear optics; (190.5530) Pulse propagation and solitons.

References and links

1. P. G. Kevrekidis, K. Ø. Rasmussen, and A. R. Bishop, "The Discrete Nonlinear Schrödinger Eauation: a Survey of Recent Results," Int. J. mod. Phys. B **15,** 2833–2900 (2001).
2. A. A. Sukhorukov, Yu. S. Kivshar, H. S. Eisenberg, and Y. Silberberg, "Spatial optical solitons in waveguide arrays," IEEE J. Quantum Electron. **39,** 31–50 (2003).
3. E. Trias, J. J. Mazo, and T. P. Orlando, "Dicrete breathers in nonlinear lattices: experimental detection in Josephson array," Phys. Rev. Lett. **84,** 741–744 (2000).
4. U. T. Schwarz, L. Q. English, and A. J. Sievers, "Experimental generation and observation of intrisic localized spin wave modes in an antiferromagnet," Phys. Rev. Lett. **83,** 223–226 (1999).
5. P. J. Y. Louis, E. A. Ostrovskaya, and Yu. S. Kivshar, "Dispersion control for matter waves and gap solitons in optical superlattices," J. Opt. B: Quantum Semiclass. Opt. **6,** S309-S317 (2004).
6. Yu. S. Kivshar and B. Luther-Davies, "Dark optical solitons: physics and applications," Phys. Rep. **298,** 81–197 (1998).
7. B. Sanchez-Rey and M. Johansson, "Exact numerical solutions for dark waves on the discrete nonlinear Schrödinger equation," Phys. Rev. E **71,** 036627(2005).
8. E. Smirnov, C. E. Rüter, M, Stepić, D. Kip, and V. Shandarov, "Formation and light guiding properties of dark solions in one-dimensional waveguide arrays," Phys. Rev. E **74**, 065601(R) (2006).
9. E. P. Fitrakis, P. G. Kevrekidis, H. Susanto, and D. J. Frantzeskakis, "Dark solitons in discrete lattices: Saturable versus cubic nonlinearitis," arXiv:nlin.PS/0608023 (2006).
10. T. R. O. Melvin, A. R. Champneys, P. G. Kevrekidis, and J. Cuevas, "Radiationless traveling waves in saturable nonlinear Schrödinger lattices," Phys. Rev. Lett. **97,** 124101 (2006).
11. Lj. Hadžievski, A. Maluckov, M. Stepić, and D. Kip, "Power controlled soliton stability and steering in lattices with saturable nonlinearity," Phys. Rev. Lett. **93,** 033901 (2004).
12. S. Wiggins, *Global Bifurcations and Chaos: Analytical Methods* (Springer-Verlag New York Inc., 1988).

1. Introduction

In the past two decades the investigation of discrete nonlinear lattices with continuous evolution variable and discrete spatial variables has exhibited a tremendous growth. It was stimulated and

#81156 - $15.00 USD Received 20 Mar 2007; revised 12 Apr 2007; accepted 13 Apr 2007; published 25 Apr 2007

triggered by the recent progress in the fabrication of nonlinear optical periodic media which allowed the experimental observation of nonlinear effects and led to the discovery of many new fundamental nonlinear/discrete phenomena [1, 2]. However, these phenomena are universal and relevant for many non-optical systems, as localized voltage drops in ladders of the Josephson junctions [3], localized modes in the antiferromagnetic crystals [4], or localization of matter waves in the Bose - Einstein condensates using optically-induced periodic potentials [5]. A special attention is devoted to the discrete intrinsic localized modes like discrete solitons and breathers which are candidates for guiding, steering, and switching of light beams in the nonlinear optical lattices. The most studies are about bright localized structures in the lattices with different type of nonlinearity [1]. However, during the last several years the interest in the dark localized structures has been increased and many papers are published mainly considering the lattices with Kerr nonlinearity [6, 7]. Very recently, dark solitons were experimentally observed in the defocusing lithium niobate waveguide arrays with saturable nonlinearity [8]. The authors demonstrated stable propagation of the dark inter-site mode. These results were discussed in [9] with a conclusion that the dark inter-site modes are unstable but with a weak instability growth rate which explains their experimental observation. The aim of this paper is to present a detail analytical and numerical study of the problems of existence, stability, and transverse motion of the dark localized modes which will give a better insight in this phenomena. We present analysis of the dark soliton existence and linear stability and investigate the transverse motion of the dark localized modes adopting the concept of the Peierls-Nabarro potential. All results are verified numerically and commented with the respect to the recent results [8, 9].

2. Stability analysis

We start our study with the one-dimensional discrete nonlinear Schrödinger (DNLS) lattice model with saturable nonlinearity given by [10, 11]

$$i\frac{\partial U_n}{\partial t} + U_{n+1} + U_{n-1} - 2U_n + \gamma\frac{U_n}{1+|U_n|^2} = 0, \tag{1}$$

where U_n is the normalized wave function in the n-th lattice element $(n = 1,2...)$ and γ is the nonlinearity parameter. For $\gamma > 0$ the nonlinearity is defocusing (DF). Under the transformation $U_n(t) = \exp(i\pi n)\exp(-4it)V_n^*(t)$ the equation (1) is mapped into the same DNLS equation for V_n where $\gamma < 0$ corresponds to the self-focusing (SF) nonlinearity. The Eq. (1) represents a system of linearly coupled nonlinear difference-differential equations which are not integrable in general case but posses two conserved quantities, Hamiltonian $H = \sum_n [-\gamma \ln(1+|U_n|^2) + |U_{n-1} - U_n|^2]$ and norm (power) $P = \sum_n |U_n|^2$.

The model equation supports localized solutions of various types [1, 2]. The bright unstaggered localized modes (solitons, breathers) can exist in systems with SF nonlinearity and the bright staggered modes in systems with DF nonlinearity. Here, we study the problem of the existence and stability of the dark localized modes in a form of localized dips (holes) on the lattice background with a phase shift across the localizing region [2]. Because the results for dark localized modes in systems with DF and SF nonlinearities are qualitatively the same, only the results for DF case [8] are presented for the sake of simplicity.

The starting point is the analysis of the continuous wave (CW) solutions of Eq. (1) which represent actual background of the dark localized modes. Generally, the assumption $U_n(t) = \phi_n e^{i\omega t}$, where ω is the propagation parameter, leads to the steady state version of Eq. (1)

$$-\omega\phi_n + \phi_{n+1} + \phi_{n-1} - 2\phi_n + \gamma\phi_n/(1+\phi_n^2) = 0. \tag{2}$$

For the CW solution the envelope ϕ_n is independent on n and we can assume $\phi_n = U_c$. With this

#81156 - \$15.00 USD Received 20 Mar 2007; revised 12 Apr 2007; accepted 13 Apr 2007; published 25 Apr 2007

assumption we obtain the following solution

$$U_c^2 = (\gamma - \omega)/\omega. \tag{3}$$

To consider the linear stability of the stationary modes [1, 7] we introduce small complex perturbations ε_n to the wave envelopes in a form $U_n(t) = (\phi_n + \varepsilon_n(t))e^{i\omega t}$. After short and simple algebraic procedure the linearized equation for the small perturbations ($\varepsilon_n << \phi_n$) is obtained

$$i\frac{\partial \varepsilon_n}{\partial t} - (\omega+2)\varepsilon_n + \varepsilon_{n+1} + \varepsilon_{n-1} + \frac{\gamma}{(1+\phi_n^2)^2}\left(\varepsilon_n - \phi_n^2\varepsilon_n^*\right) = 0. \tag{4}$$

By splitting ε_n into the real and imaginary part $\varepsilon_n = f_n + ig_n$, the evolution of the perturbation is described by the the system of two equations for the real functions. The system written in a matrix form reads

$$\frac{d}{dt}\begin{bmatrix} f_n \\ g_n \end{bmatrix} = \begin{bmatrix} 0 & H^+ \\ -H^- & 0 \end{bmatrix}\begin{bmatrix} f_n \\ g_n \end{bmatrix} \equiv \mathbf{M}\begin{bmatrix} f_n \\ g_n \end{bmatrix}, \tag{5}$$

where the matrix $\mathbf{M}$, which is generally non-hermitian [7], for the lattice with N elements has dimension $2N \times 2N$. The submatrices $H^{\pm}$ $(N \times N)$ can be written in the explicit form

$$H_{ij}^+ = (\omega+2)\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1} - \frac{\gamma}{1+\phi_n^2}\delta_{ij}, \quad H_{ij}^- = H_{ij}^+ + 2\gamma\frac{\phi_n^2}{(1+\phi_n^2)^2}\delta_{ij}, \tag{6}$$

where δ_{ij} is the Kroneker symbol.

For the CW solution, assuming perturbations in a form $(f_n, g_n) = (f, g)e^{\Omega t}e^{iK_p n}\cos(qn)$, where $K_p = \pi$ for staggered, $K_p = 0$ for unstaggered perturbations and q is the wave number, the following general dispersion relation is obtained

$$\Omega^2 = -4\left[\gamma(1+\cos K_p \cos q + \omega) - \omega^2\right]/\gamma. \tag{7}$$

In general, analyzing the condition for the modulation instability ($\Omega^2 > 0$) for different CW solutions of the DNLS equation (1) [one of them is written in (3)] we can conclude: a) For DF nonlinearity the CW staggered solutions are unstable which gives a possibility for creation of the staggered bright solitons, while the CW unstaggered solutions are stable which provides a stable background for creation of the unstaggered dark solitons; b) For SF nonlinearity the CW unstaggered solutions are unstable which gives a possibility for creation of the unstaggered bright solitons, while the CW staggered solutions are stable which provides a stable background for creation of the staggered dark solitons.

3. Dark modes

For the unstaggered dark solitons two different configurations can exist: the on-site, with a gap centered on the lattice element and inter-site, with a gap centered between two neighboring lattice elements. Schematically the patterns for two types of dark solitons can be represented as: $(...1,1,1,0,-1,-1,-1...)$ - on-site and $(...1,1,1,-1,-1,-1...)$ - inter-site. The linear stability analysis for dark soliton solutions is based on the properties of the eigenvalues of the corresponding matrix M. The eigenvalue (EV) spectrum has contributions from two sources. One is a continuous part of EV spectrum which arises from the background. The corresponding EV functions are plane waves. For the CW the dispersion relation (7) gives pure imaginary eigenvalues ($\Omega^2 < 0$) and consequently does not indicate instability of dark solitons. The second source is associated with the central part of the dark soliton configuration and represents the discrete part of the EV spectrum. The discrete EV spectrum by itself and through the interaction

#81156 - $15.00 USD Received 20 Mar 2007; revised 12 Apr 2007; accepted 13 Apr 2007; published 25 Apr 2007

with the continuous part of the EV spectrum can be associated with the eventual instability of the dark solitons. For this reason we will focus our study to the discrete spectrum.

The first estimates of the discrete spectrum can be done using approximate dark soliton solutions described simply as a corresponding CW solution with a phase inversion at the n_c element. These solutions can be obtained by multiplying the unstaggered patterns with the amplitudes of the corresponding CW solution (3). The approximation holds in the limit of large amplitudes which corresponds to the values of ω near the left boundary of the existence domain. Substitution of the approximate dark soliton solutions into the Eq. (5) allows calculation of the discrete eigenvalues

$$\Omega_{uo} = \pm i\sqrt{(\gamma-\omega-2)^2+2},\ \Omega_{ui} = \pm i\sqrt{2(1+\omega-\omega^2/\gamma)}, \tag{8}$$

where Ω_{uo} and Ω_{ui} are discrete eigenvalues of the on-site and inter-site unstaggered configurations, respectively. These expressions show that the discrete eigenvalues are pure imaginary in the region of the dark soliton existence ($0 < \omega < \gamma$) and does not give indication of the soliton instability. However, it is not a proof of the soliton stability because the eigenvalues with finite real part may appear with the introduction of the exact soliton solutions in calculations.

The full spectrum of $2N$ complex eigenvalues of $\mathbf{M}$ is found numerically for a different number of lattice elements N and different values of the parameters γ and ω. Generally, for both on-site and inter-site configurations we can observe a subset with pure imaginary eigenvalues embedded in the continuous part of the EV spectrum (Fig. 1; the shaded regions), which is well described with the dispersion curve (7). The density of the eigenvalues inside the shaded region increases with the increase of N, approaching the continuum when $N \to \infty$. However, a discrete part of the EV spectrum where eigenvalues with a positive real part exist for some intervals in ω indicates instability of the system.

For the on-site configuration there are branches of the discrete EV spectrum with complex eigenvalues which indicate presence of the oscillatory instability. With the change of parameter values four complex EVs appear, exist and merge after which only branches with pure imaginary EVs remain in EV spectrum. Therefore, taking formally ω as a bifurcation parameter the bifurcations of the Hopf type [12] for on-site dark modes are indicated. The bifurcation points coincide with the intersection of the discrete and continuous part of the EV spectrum (Fig. 1). The analytically calculated discrete branch for the on-site configuration (8) is in good agreement with the numerical results in the region near the left boundary of the existence region (small ω) where approximation used for the analytical calculations is valid.

For the inter-site configuration numerical results show existence of branches of the discrete spectrum with pure real eigenvalues which indicates presence of the exponentially growing instability (Fig. 1). Taking ω as a bifurcation parameter the merge of two purely real EVs after which only pure imaginary EVs remain in discrete spectrum is formally noted as a tangential (saddle-center) bifurcation [12]. However, the analytically calculated discrete branch for the inter-site configuration (8) near the left boundary of the existence region (small ω) is embedded in the continuous part of the spectrum which is not consistent with the numerical results. In this case the bifurcation appears only near the upper boundary of the existence region (high ω - small amplitudes) where approximate solution (8) is not valid. The correct approach in this region is a calculation of the discrete spectrum in the limit $U_c << 1$, which gives the expression consistent with the numerical results: $\Omega_{ui} = \pm i\sqrt{(\gamma-\omega-2)^2+(\gamma-\omega-2)+2\omega(\omega/\gamma-1)}$. The zero of this expression approximately corresponds to the bifurcation point ω_b and indicates existence of the pure imaginary branch ($\omega > \omega_b$) associated with the existence of the neutrally stable dark inter-site breathers (Fig.1).

Finally, we can conclude that all configurations of the dark solitons are unstable in the most part of the existence region. However, there are also regions where only EVs with $Re(\Omega) \approx 0$

#81156 - $15.00 USD Received 20 Mar 2007; revised 12 Apr 2007; accepted 13 Apr 2007; published 25 Apr 2007

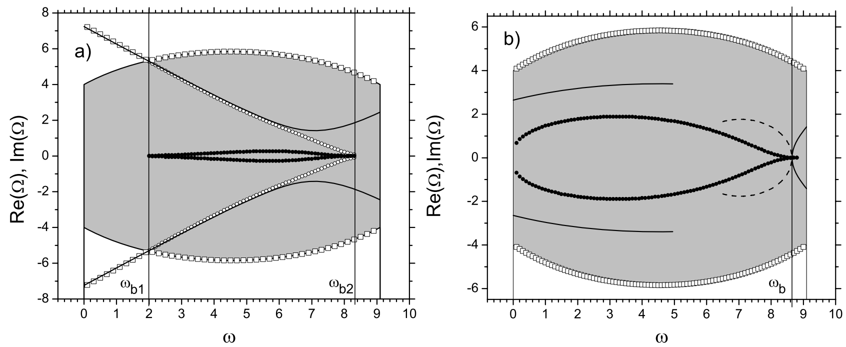

Fig. 1. Eigenvalues spectrum for the on-site (a) and the inter-site (b) dark localized mode. Numerical results are given with symbols, analytical with lines: squares (solid lines) for extremal imaginary EV, solid circles (dashed lines) for the real and circles for the imaginary part of EV discrete spectrum. The continuous EV spectrum is displayed as a shaded region.

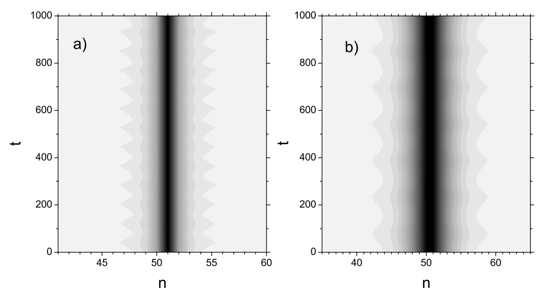

Fig. 2. Illustration of the stable propagation of the unstaggered dark breathers: a) the on-site near the left boundary of the existence region, $\omega = 0.08$ and b) inter-site near the right boundary of the existence region, $\omega = 8.9$.

and $Im(\Omega) \neq 0$ exist which indicates neutral stability and possibility for existence of the dark localized structures of the breather type. These regions for the on-site dark breather configurations are near the lower ($\omega < \omega_{b1}$, large amplitudes) and upper ($\omega > \omega_{b2}$, small amplitudes) boundaries of the existence region (Fig.1). The inter-site dark breather configurations can exist only in the region near the upper boundary of the existence region ($\omega > \omega_b$, small amplitudes). These results are confirmed numerically directly solving the model equation (1) as a Cauchy problem with initial conditions in a form of slightly perturbed dark solitons. The time-space evolution shows existence of the stable dark breathers in the predicted regions (Fig. 2) and instability in all other cases. This is consistent with the recently published experimental results [8] where the experimental observation of the dark on-site and inter-site modes in defocusing lithium niobate waveguide arrays are reported. However, we also observe propagation of long-lived dark localized structures in the instability region with the weak instability growth rates which agrees with the result in [9].

4. Transversal motion

In order to study the transverse motion of dark breathers across the lattice, instead of the conserved quantities P and H, which diverge for the localized dark configuration we use the com-

#81156 - $15.00 USD Received 20 Mar 2007; revised 12 Apr 2007; accepted 13 Apr 2007; published 25 Apr 2007

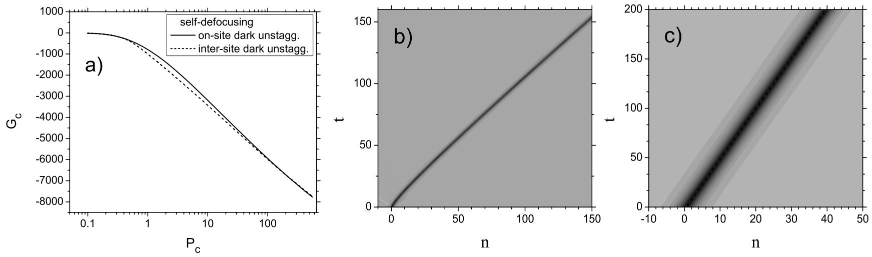

Fig. 3. The grand canonical free energy of the on-site and inter-site dark soliton (a). Free transverse motion of the discrete dark breathers for b) $\omega = 0.08$ and c) $\omega = 8.9$.

plementary quantities P_c and H_c, where the Bloch-wave background (U_{cw}) is removed [1, 2]

$$P_c = \sum_n (U_{cw}^2 - U_n^2),\; H_c = H + \gamma[P_c/(1+U_{cw}^2) - N\ln(1+U_{cw})].$$

Then the grand canonical free energy [10] is defined as $G_c = H_c - \omega P_c$.

The energy difference ΔG_c between on-site and inter-site dark soliton configurations with the same norm P_c has a sense of the potential barrier which arises from the discreteness of the system and can be taken as a measure of the well-known Peierls-Nabarro (PN) barrier. The $G_c(P_c)$ for the unstaggered on-site and inter-site dark soliton configurations is displayed in Fig. 3. The remarkable feature of stability alternation between on-site and inter-site configurations observed for the bright solitons in lattices with SF nolinearity is absent [11]. There are no discrete transparent points which correspond to the zeros of the ΔG_c. Instead, as can be clearly seen, the energy difference $\Delta G_c(P_c)$ for $P_c << 1$ and $P_c >> 1$ vanishes. These regions coincide with the regions where our previous stability analysis predicted the existence of the dark breather configurations. These facts lead to the significant conclusion that the dark breathers in these regions are not affected by the PN barrier and can freely move across the lattice elements. As a consequence, we expect and confirm numerically that a small phase perturbation of any dark soliton configuration (on-site or inter-site) will cause creation of the moving dark breather (Fig. 3). This behavior indicates possibility for an easy experimental observation of the moving dark breathers with a similar experimental setup as in [8]. All results and conclusions about transversal motion remain qualitatively the same for the staggered dark breathers in a system with SF nonlinearity.

5. Conclusion

In conclusion, we point out few significant results. The stability analysis indicates possibility of the existence of stable on-site and inter-site dark breathers in lattices with saturable nonlinearity which is confirmed numerically. This is found to be consistent with the recent experimental observations [8]. The analysis based on the complementary conserved quantities indicates the existence of regions with the zeroth value of the PN barrier where transversal motion of the dark breathers is possible. The numerical results confirm this expectation and indicate possibility for its experimental confirmation.

Acknowledgments

Support from the Ministry of Science and Environmental Protection of Serbia is acknowledged.